INTEGRAL INVARIANTS OF TWO-DIMENSIONAL ROTATIONAL GAS FLOWS

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In a previous work [1] we have constructed the functional

$$J = \iint f(x, y, \psi, \psi_x, \psi_y)\, dx\, dy$$

for which Euler's equation coincides with Crocco's equation, which describes the two-dimensional rotational flow of a gas. In this paper we shall apply group theoretical methods to the study of this functional and find a group of transformations which leave this functional invariant.

Before we proceed to analyze the functional, we note the following property. It is known that the functional is determined by the given equation to within divergence-type terms. Since these terms do not affect Euler's equation, they can be disregarded. Clearly, if the functional admits a certain transformation group G_1, that the group will automatically leave the corresponding Euler's equation invariant. In what follows we shall see that that group is, in general, a subgroup of the group G_a, associated with the corresponding Euler's equation. The group properties of an integral were first investigated by Lie, but a general method of obtaining conservation laws was given in Noether's work [2]. A simple and detailed derivation of Noether's theorem (for a special case) is given in [3]. Since the theory of integral invariants is a special case of the theory of differential invariants, all of the theory developed in [4] can be extended to integrals. The derivations in this paper apply to an integral of arbitrary form

$$J = \iint \ldots \int f\left(x,\ u,\ \frac{\partial u}{\partial x}, \ldots, \frac{\partial^r u}{\partial x^r}\right) dx_1 \ldots dx_n.$$

Consider the functional

$$J = \iint \rho_0(\psi)(1 - V^2)^{1/\gamma - 1}\left(1 + \frac{\gamma + 1}{\gamma - 1} V^2\right) dx\, dy = \iint \rho_0(\psi) F(V^2)\, dx\, dy = \iint f(x, y, \psi, \psi_x, \psi_y)\, dx\, dy. \qquad (1)$$

Here ψ is the stream function, V is the speed, γ is the adiabatic exponent, ρ_0 is the stagnation density, and the rest of the notation is obvious. We now proceed to construct a group which leaves the integral (1) invariant. We shall consider the group of point transformations

$$x^* = f^1(x, y, \psi), \qquad y^* = f^2(x, y, \psi), \qquad \psi^* = f^3(x, y, \psi).$$

We shall call the functional (1) invariant with respect to this group if

$$\iint\limits_{D(x)} f(x, y, \psi, \psi_x, \psi_y)\, dx\, dy = \iint\limits_{D(x^*)} f(x^*, y^*, \psi^*, \psi_{x^*}{}^*, \psi_{y^*}{}^*)\, dx^*\, dy^*. \qquad (2)$$

Let the operator of the one-parameter subgroup

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial \psi}$$

belong to the group of transformations which leave the functional (1) invariant. The extended operator X^* has the form

$$X^* = X + \eta^1 \frac{\partial}{\partial p} + \eta^2 \frac{\partial}{\partial q}, \quad p = \psi_x, \quad q = \psi_y.$$

The necessary condition for the invariance of the functional (1) with respect to the operator X is

$$X^* f + f(\xi_x{}^1 + p\xi_\psi{}^1 + \xi_y{}^2 + q\xi_\psi{}^2) = 0. \qquad (3)$$

This condition is derived in the following way. The operator X corresponds to variable transformation, which is close to the unit transformation, of the one-parameter subgroup

$$x^* = x + t\xi^1, \qquad y^* = y + t\xi^2, \qquad \psi^* = \psi + t\xi^3,$$
$$p^* = p + t\eta^1, \qquad q^* = q + t\eta^2$$

(t is a parameter). Substituting these values in (1) and taking account of terms linear in t only, we obtain

$$J^* - J = \iint\limits_{D(x)} [X^* f + f(\xi_x{}^1 + p\xi_\psi{}^1 + \xi_y{}^2 + q\xi_\psi{}^2)]\, dx\, dy.$$

Since this equality holds for an arbitrary domain, this yields condition (3). The sufficiency of this condition is proved, for example, in Noether's work [2].

For the functional (1) $f_x = f_y = 0$, so that (3) becomes

$$\xi^3 f_\psi + \eta^1 f_p + \eta^2 f_q + f(\xi_x{}^1 + \xi_\psi{}^1 p + \xi_y{}^2 + q\xi_\psi{}^2) = 0. \qquad (4)$$

From (1) we find

$$f_\psi = \rho_0' F, \quad f_p = \frac{2\gamma}{\gamma - 1} \rho_0 p (1 - V^2)^{1/(1-\gamma)},$$
$$f_q = \frac{2\gamma}{\gamma - 1} q (1 - V^2)^{1/(1-\gamma)}. \qquad (5)$$

The coefficients η^i of the extended operator are given by the formulas [4]

$$\eta^1 = \xi_x{}^3 + x\xi_\psi{}^3 - p(\xi_x{}^1 + p\xi_\psi{}^1) - q(\xi_x{}^2 + q\xi_\psi{}^2),$$
$$\eta^2 = \xi_y{}^3 + q\xi_\psi{}^3 - p(\xi_y{}^1 + q\xi_\psi{}^1) - q(\xi_y{}^2 + q\xi_\psi{}^2). \qquad (6)$$

Substituting (5) and (6) in (4), we find

$$\xi^3 \frac{d \ln \rho_0}{d\psi} + \xi_x{}^1 + \xi_y{}^2 + p(A\xi_x{}^3 + \xi_\psi{}^1) + q(A\xi_y{}^3 + \xi_\psi{}^2) -$$
$$- pqA(\xi_x{}^2 + \xi_y{}^1) + p^2 A(\xi_\psi{}^3 - \xi_x{}^1) + q^2 A(\xi_\psi{}^3 - \xi_y{}^2) -$$
$$- p^2 q A\xi_\psi{}^2 - pq^2 A\xi_\psi{}^1 = 0,$$
$$\left(A = \frac{2\gamma}{(\gamma - 1) F(V^2)} (1 - V^2)^{1/(1-\gamma)}\right). \qquad (7)$$

Since the coefficients ξ^1, ξ^2, ξ^3 are independent of pq, we obtain the following system of equations for the variables ξ^i by equating to zero the coefficients of various powers of p, q:

$$\xi^3 \frac{d \ln \rho_0}{d\psi} = -(\xi_x{}^1 + \xi_y{}^2), \qquad (8)$$

$$\xi_x{}^3 = \xi_\psi{}^1, \quad \xi_y{}^3 = \xi_\psi{}^2, \quad \xi_x{}^2 + \xi_y{}^1 = 0,$$
$$\xi_\psi{}^3 - \xi_x{}^1 = 0, \quad \xi_\psi{}^3 - \xi_y{}^2 = 0. \qquad (9)$$

Thus, the coefficients ξ^i of the operator X are determined by the system (8), (9). When group-theoretical methods are applied to differential equations, there often arises the question of a special choice of the parametric functions, which may have an effect on the width of the group admitted by the given system of equations [4]. A quite analogous question arises in our case. Our functional (1) has, in general, two arbitrary functions: ρ_0 and $F(V^2)$. The choice of the function ρ_0 is connected with the choice of the distribution of entropy among the particles, i.e., with a definite class of rotational flows, which admits a wider group of transformations. The function $F(V^2)$ can also be regarded as arbitrary, although we restrict ourselves to the case when it is assumed to be fixed and does not enter the system of governing equations.

From the first two equations of (9) it is clear that

$$\xi^1 = \xi^1(x, y), \qquad \xi^2 = \xi^2(x, y), \qquad \xi^3 = \xi^3(\psi). \qquad (10)$$

Since ξ^3 does not depend on x, y, and ξ^1 and ξ^2 do not depend on ψ, it follows from (8) that the right and left sides are equal to a constant. On the other hand, from the last two equations of (9) it follows that $\xi^1_x + \xi^2_y = 2\xi^3_\psi$, which finally yields

$$\xi^3 \frac{d \ln \rho_0}{d\psi} = k, \quad \frac{d\xi^3}{d\psi} = -\frac{k}{2} \quad (k = \text{const}) \text{ (in particular } k = 0). \quad (11)$$

We first consider the case k = 0. This condition can be satisfied when

$$\text{(a) } \rho_0 = \text{const}, \quad \xi^3 = \text{const}, \quad \text{(b) } \xi^3 = 0, \quad \rho_0 = \rho_0(\psi).$$

Here $\rho_0(\psi)$ is a function of ψ. Consider case (b), which corresponds to rotational flows. The last two equations in (9) yield

$$\xi^1 = \xi^1(y), \quad \xi^2 = \xi^2(x), \quad \xi^3 = 0.$$

From the third equation in (9) we obtain

$$\xi^1 = ay + c_1, \qquad \xi^2 = -ax + c_2.$$

The final transformations are then

$$x^* = x\cos\omega - y\sin\omega + d_1,$$
$$y^* = x\sin\omega + y\cos\omega + d_2, \quad \psi^* = \psi. \quad (12)$$

Thus, in the general case of rotational flows the functional (1) is invariant with respect to the group (12). Clearly, this also holds with respect to Crocco's equation. The meaning of the group (12) is obvious.

The case (a) can be treated in a quite analogous way. This case corresponds to rotationless flows. The final transformation group is then

$$x^* = x\cos\omega - y\sin\omega + d_1,$$
$$y^* = x\sin\omega + y\cos\omega + d_2, \quad \psi^* = \psi + d_3. \quad (13)$$

Thus we have shown that of all two-dimensional flows the potential flows have the widest group. This, apparently, is associated with the fact that these flows are the most simple. We shall now consider the case $k \neq 0$ and show that in this case there exist rotational flows which admit a wider group than (12). We differentiate the first equation in (11) with respect to ψ and then, using these equations, we obtain

$$\frac{d^2 \ln \rho_0}{d\psi^2} - \frac{1}{2}\left(\frac{d \ln \rho_0}{d\psi}\right)^2 = 0. \quad (14)$$

This has the solution

$$\rho_0 = \frac{d_0}{(\psi + d_{00})^2}. \quad (15)$$

Consider again the functional (1). Introduce the new variables

$$x^* = a_0 x, \quad y^* = a_0 y, \quad \psi^* = a_0\psi + b_0.$$

This transforms the functional (1) into a functional of the same form with the function $\rho_{10}(\psi^*) = a_0^2\rho_0(\psi)$.

We shall call two functionals equivalent if the ρ_0 functions satisfy the relation

$$\rho_{10}(\psi^*) = a_0^2\rho_0(\psi).$$

Note that if an integral J is invariant with respect to a group G, then cJ (c = const) is also invariant with respect to that group. This means that f (and consequently ρ_0) is defined to within a constant factor. Taking account of this remark and of the equivalence transformation, we reduce (15) to the form

$$\rho_0(\psi) = \frac{1}{\psi^2}. \quad (16)$$

Now from (11) we obtain $\xi^3_\psi = -(1/2)k$. Let us denote the constant $-k/2$ by λ. The last two equations in (9) yield

$$\xi^1 = \lambda x + \varphi^1(y), \quad \xi^2 = \lambda y + \varphi^2(x), \quad \xi^3 = \lambda\psi.$$

The third equation in (9) yields $\varphi^2_x = -\varphi^1_y$. Consequently, the functions φ^i have the form

$$\varphi^1 = -A_0 y + a_1, \quad \varphi^2 = A_0 x + a_2.$$

Finally, we find the following expressions:

$$\xi^1 = \lambda x - A_0 y + a_1, \quad \xi^2 = \lambda y + A_0 x + a_2, \quad \xi^3 = \lambda\psi.$$

The coefficient λ can be taken equal to one, since the operator X is defined to within a constant multiplier. The final transformations are

$$x^* = cx\cos\omega - cy\sin\omega + d_1, \quad y^* = cx\sin\omega + cy\cos\omega + d_2,$$
$$\psi^* = c\psi. \quad (17)$$

The identity transformation corresponds to $\omega = d_1 = d_2 = 0$, $c = 1$. Thus, for the values of ρ_0 which correspond to (16) we obtain a four-termed group of transformations. Group (17) contains, in addition to translations and rotations in the x, y plane, also stretching transformations. For given values of ρ_0, we can easily find from (16) the entropy distribution $S = \text{const} \cdot \ln\psi$. Group (16), as well as group (13), contain four parameters.

It is known [2, 3] that the existence of a r-parameter group for an integral automatically yields r divergence-type relations. When the integral has only one independent variable, these relations yield r first integrals. The existence of the three-parameter group (12) for the integral (1) allows us to write down these divergence-type relations. Two of these, as can be easily verified, yield the law of conservation of momentum, and the third relation follows from the first two.

The invariance of the functional (1) with respect to the operator X makes it possible to write down ordinary differential equations, which can also be obtained from Crocco's equation. Consider, for example, the operator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial\psi},$$

which corresponds to $\omega = d_1 = d_2 = 0$ (cf. (17)). Its invariants are

$$J^1 = x/y = \vartheta, \qquad J^2 = \psi/x.$$

Consequently, we can look for a solution in the form $\psi = x\varphi(\vartheta)$. The substitution of this expression in Crocco's equation yields the ordinary differential equation

$$\vartheta\varphi''\left[\left(1 - \frac{u^2}{a^2}\right) + \frac{2uv}{a^2}\vartheta + \left(1 - \frac{v^2}{a^2}\right)\vartheta^2\right] = \frac{(1 - M^2)}{2\gamma gR\vartheta\varphi(\vartheta)}(1 - V^2)^{\gamma+1/\gamma-1}.$$

The solution of this equation represents a rotational flow with velocity constant along a ray. Let us assume, formally, that the right side equals zero. The left side is a product of two factors. Equating the first of these to zero we obtain a uniform flow, and the second yields a Prandtl-Meyer flow. In fact, taking the x axis to be in the direction of the velocity vector, we have u = V, v = 0, the second factor yields $\vartheta = \sqrt{M^2 - 1}$, and the projection of the velocity on the ray is

$$\frac{V}{1 + \vartheta^2} = a.$$

Consider now the relation between the groups G_i and G_a. Since these relations can take many forms, we shall discuss only the most characteristic cases. The inclusion $G_i \subseteq G_a$ is obvious. We shall consider the set $G_a - G_i$ and shall find out what type of transformations it includes.

1) The most common case is that in which the set $G_a - G_i$ consists of transformations which give a multiplier in front of the integral during the transformation of the functional. If the functional

$$J = \int f(x, y, y')\,dx$$

is transformed by the transformation $x^* = \varphi^1(x, y)$, $y^* = \varphi^2(x, y)$ into the functional

$$J^* = \lambda\int f(x^*, y^*, y^{*\prime})\,dx^*,$$

where λ is the group multiplier, then it is obvious that this transformation does not belong to G_i, but does belong to G_a.

As an example, consider Crocco's equation with S = const. In this case it admits, as can be easily verified, the transformation $x^* = \lambda x$, $y^* = \lambda y$, $\psi^* = \lambda\psi$. The corresponding functional, however,

$$J = \iint F(V^2)\,dx\,dy,$$

is transformed by this transformation into

$$J^* = \frac{1}{\lambda^2}\iint F(V^{*2})\,dx^*\,dy^* = \frac{1}{\lambda^2}J,$$

and the condition $J^* = J$ is not satisfied. Thus, these transformations are lost in the transition from the equation to the functional.

2) Another common case is that in which Euler's equation admits an infinite group. We shall consider three examples.

a) The functional

$$J = \iint (\psi_x^2 + \psi_y^2)\,dx\,dy.$$

Clearly, this example corresponds to Case (1).

b) Chaplygin's equation with the corresponding functional. It is known [4] that if one disregards such trivial transformations as the multiplication of the solution by a constant or the addition to the solution of any other solution of Chaplygin's equation, G_a is then either a one-parameter or a three-parameter group. The corresponding functional also gives a one- or three-parameter group. Thus, if we exclude all trivial transformations, we obtain $G_a = G_i$.

c) The functional

$$\int \sqrt{1 + y'^2}\,dx$$

with the obvious linear Euler's equation $y'' = 0$. The extremals of this functional are straight lines. The projective transformations

$$x^* = \frac{ax + by + c}{a_0x + b_0y + c_0}, \qquad y^* = \frac{a_1x + b_1y + c_1}{a_0x + b_0y + c_0}$$

transform straight lines into straight lines and, consequently, transform the equation $y'' = 0$ into $y^{*''} = 0$. These projective transformations belong to the group G_a, whereas the group G_i for the functional consists of the translations in the x, y plane.

3) Finally, there is a third case, in which the transformation from x, y, ψ to x^*, y^*, ψ^* transforms the quadratic form $\omega = f(x, y, \psi, \psi_x, \psi_y)dxdy$ into the form

$$\omega^* = \lambda f(x^*, y^*, \psi^*, \psi_{x^*}^*, \psi_{y^*}^*)\,dx^*dy^* + \mu\,\mathrm{div}\,B(x^*, y^*, \psi^*)\,dx^*dy^*,$$

where λ, μ are group multipliers and $B(\ldots)$ is some function. Clearly, such transformations belong to the set G_a, since Euler's equation is not changed by the addition of divergence-type terms, but it does not belong to the set G_i.

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